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LETTER TO THE EDITOR

Mirror theory of spin systems with a surface

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Abstract. We discuss the mirror theory of spin systems with a surface. In the semi-infinite system, the 2-point correlation function at bulk critical temperature depends only on the 'real' distance and the 'image' distance, as was shown by the present authors within the framework of $1/n$ expansion and more recently by Cardy with the use of the conformal invariance. According to this mirror theory, we can directly show the scaling relation $2\eta_\perp - \eta_\parallel = \eta$. We also find a universal combination of amplitudes.

We check the mirror theory by means of the $\varepsilon(=4-d)$ expansion and present the explicit form of the correlation function in real space up to $O(\varepsilon^2)$. The resulting surface critical exponents $\eta_\perp$ and $\eta_\parallel$ coincide with those obtained previously.

The critical phenomena at surfaces have currently aroused considerable interest (see Binder (1983) and references quoted therein). In spin systems with a surface, an understanding of the role of the surface is important in order to determine the correlation functions. Investigating the critical behaviour of the ordinary and special transitions in the framework of $1/n$ expansion, Ohno and Okabe (1983) pointed out that the surface of the semi-infinite system plays the role of 'mirror' at the bulk critical temperature. This mirror theory imposes some constraint for the 2-point correlation function $G(r_1, r_2)$ at the bulk critical temperature. That is, only the 'real' distance $r=[\rho^2+(z_1-z_2)^2]^{1/2}$ and the 'image' distance $\bar{r}=[\rho^2+(z_1+z_2)^2]^{1/2}$ become relevant lengths at criticality, where z_1 and z_2 are normal distances from the surface and ρ is a parallel distance projected onto the surface (see figure 1). In other words, together with the requirement of a homogeneous function, the 2-point correlation function is usually written as

$$G(r_1, r_2) = G_\rho(z_1, z_2) = G^{\text{bulk}}(r)\Psi(r/\bar{r}). \tag{1}$$

Here $G^{\text{bulk}}(r) = Cr^{2-d-\eta}$ denotes the bulk correlation function (d is the spatial dimension and η the bulk anomalous dimension) and $\Psi(r/\bar{r})$ is an unknown function of $r/\bar{r}$ which goes to unity as $r/\bar{r}$ goes to zero. Equation (1) can be rewritten in a slightly different form:

$$G_\rho(z_1, z_2) = (z_1 z_2)^{(2-d-\eta)/2} f(v) \tag{2}$$

with a non-dimensional argument

$$v = \frac{z_1^2 + z_2^2 + \rho^2}{2 z_1 z_2} = \frac{\bar{r}^2 + r^2}{\bar{r}^2 - r^2}. \tag{3}$$

(See § 6 of Ohno and Okabe (1983) for further details.)

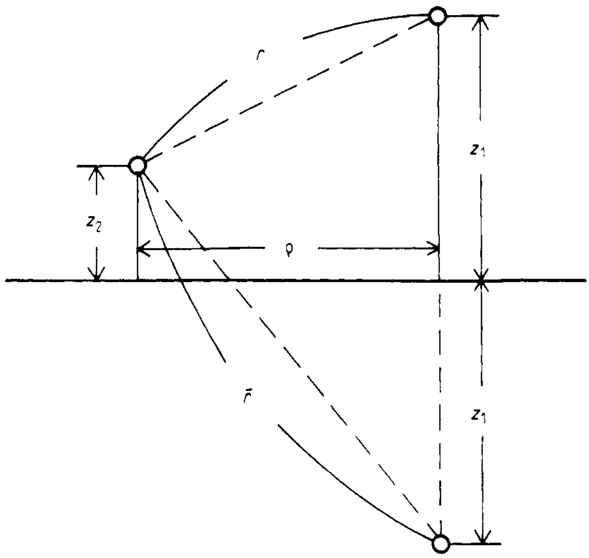

Figure 1. Geometry specifying the correlation function between two points $r_1 = (x_1, z_1)$ and $r_2 = (x_2, z_2)$. z_1 and z_2 are normal distances from the surface and $\rho = |x_1 - x_2|$ is a distance parallel to the surface. The correlation function depends only on the real distance $r = [\rho^2 + (z_1 - z_2)^2]^{1/2}$ and the image distance $\bar{r} = [\rho^2 + (z_1 + z_2)^2]^{1/2}$.

The mirror theory holds for general semi-infinite (spin) systems in spite of their intrinsic nonlinearity. Very recently, Cardy (1984) derived equation (2) by the use of the invariance under the special conformal transformation

$$r_i'/r_i'^2 = (r_i/r_i^2) + a, \tag{4}$$

where a is an arbitrary vector parallel to the surface. This transformation has often been utilised in bulk critical phenomena (Polyakov 1970, Wegner 1976), but in surface critical phenomena for the first time by Cardy (1984). Figure 2 shows an example of the square lattice ($d = 2$) transformed by (4). Although distortions become significant away from the origin, local angles and surface geometry are preserved under such a mapping. The correlation functions are invariant under this transformation because of the local invariance of the fixed-point Hamiltonian $\mathscr{H}^*$. Cardy (1984) treated the translational vector a to be infinitesimal, but it should be emphasised that such an assumption is not necessary to prove the mirror theory. Indeed, the ratio of the real distance to the image one $r/\bar{r}$ is invariant under the transformation (4) with an arbitrary a parallel to the surface; this is shown by an elementary d-dimensional vector algebra. Thus the 2-point correlation function must take the form of equation (1) with an unknown function $\Psi(r/\bar{r})$. Previously, Lubensky and Rubin (1975) proposed a scaling form similar to ours (see equation (4.21) of their paper). However, their conjecture for the form of $\Psi(r/\bar{r})$ is too restricted and cannot be accepted in general.

As the consequence of the mirror theory, we can directly derive one of the scaling relations (Lubensky and Rubin 1975, Binder 1983)

$$2\eta_\perp - \eta_\parallel = \eta \tag{5}$$

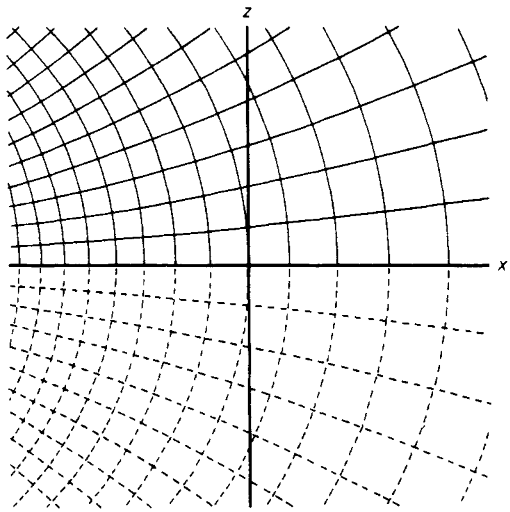

Figure 2. Square lattice transformed by the special conformal transformation (4) with $a = (-0.05, 0)$. The original axes (x and z) are shown in bold full lines, both corresponding to the interval $(-6, 6)$. Each distorted square corresponds to a unit square in the underlying coordinates. Although the distortions become significant away from the origin, local angles and surface geometry are preserved. One may use a scale to see that the ratio of the real distance to the image one $r/\bar{r}$ is invariant under the transformation.

among surface critical exponents. These critical exponents are defined by

$$G_\rho(z_1, z_2) \xrightarrow[\rho\to\infty]{} C_\parallel(z_1, z_2)/\rho^{d-2+\eta_\parallel} \tag{6}$$

$$\xrightarrow[z_1\to\infty]{} C_\perp(z_2)/z_1^{d-2+\eta_\perp}. \tag{7}$$

Suppose $f(v)$ tends to zero as $f(v) \to v^{-\tilde{\eta}}$ for large v. Then, combining this asymptotic behaviour with equations (2), (3), (6) and (7), one gets $\eta_\parallel = 2 - d + 2\tilde{\eta}$ and $\eta_\perp = \frac{1}{2}(2 - d - \eta) + \tilde{\eta}$ and in turn derives the surface scaling relation (5). Moreover, the ratio of amplitudes

$$(C_\perp(z_1)C_\perp(z_2))/(C_\parallel(z_1, z_2)C) \tag{8}$$

is shown to be universal (C is an amplitude for $G^{\text{bulk}}(r)$). Similar universal combinations of critical amplitudes have been discussed by Okabe and Ohno (1984).

Since the mirror theory does not determine the form of the function $f(v)$ completely (the only exception is the case of $d = 2$: see Cardy 1984), one should derive an explicit form of $f(v)$ or at least an equation satisfied by $f(v)$. For perturbation theories such as the ε or $1/n$ expansion, the method adopted by Ohno and Okabe (1983, 1984) is useful in determining the real space correlation function.

The rest of this letter is devoted to presenting a new result with the $\varepsilon(=4-d)$ expansion for the semi-infinite O(n) model. First of all, within a mean-field theory, the 2-point correlation function is given by

$$G_\rho^{(0)}(z_1, z_2) = (z_1 z_2)^{-1+\varepsilon/2}(2\pi)^{-d/2} g^{(0)}(v). \tag{9}$$

Here the function $g^{(0)}(v)$ is, in the $\varepsilon\to 0$ limit,

$$g^{(0)}(v)=\begin{cases}(v^2-1)^{-1} & \text{for the ordinary transition}\\ v(v^2-1)^{-1} & \text{for the special transition,}\end{cases} \tag{10}$$

which satisfies $\mathscr{D}g^{(0)}=0$ with

$$\mathscr{D}=(v^2-1)\,\mathrm{d}^2/\mathrm{d}v^2+(4-\varepsilon)v\,\mathrm{d}/\mathrm{d}v+(1-\tfrac{1}{2}\varepsilon)(2-\tfrac{1}{2}\varepsilon). \tag{11}$$

To the order of two loops, the correlation function is explicitly given in the form (2) with (3) and $f(v)=f^{(a)}(v)+f^{(b)}(v)$, where $f^{(a)}(v)$ and $f^{(b)}(v)$ satisfy

$$(\mathscr{D}-\zeta)f^{(a)}(v)=0, \tag{12}$$

$$\mathscr{D}^2f^{(b)}(v)=\eta(2\pi)^{-d/2}(g^{(0)}(v))^3, \tag{13}$$

respectively. Here, $\eta=(n+2)\varepsilon^2/2(n+8)^2+\mathrm{O}(\varepsilon^3)$ is the bulk anomalous dimension and ζ is a parameter of $\mathrm{O}(\varepsilon)$; writing $\zeta=\mu^2-\frac{1}{4}$,

$$\mu=\frac{1}{2}-\frac{n+2}{2(n+8)}\varepsilon-\frac{(n+2)(17n+76)}{4(n+8)^3}\varepsilon^2 \tag{14}$$

for the ordinary transition and

$$\mu=-\frac{1}{2}-\frac{n+2}{2(n+8)}\varepsilon-\frac{5(n+2)(n-4)}{4(n+8)^3}\varepsilon^2 \tag{15}$$

for the special transition. Solving (12) and (13) leads us to our final result

$$f(v)=\frac{1}{(2\pi)^{d/2}}\left[\exp[\tfrac{1}{2}(2-d)\pi\mathrm{i}]\frac{Q^{(d-2)/2}_{\mu-1/2}(v)}{(v^2-1)^{(d-2)/4}}\right.$$
$$\left.+\eta\frac{y}{v^2-1}\int^v\frac{\mathrm{d}v}{y}\int_\infty^v\mathrm{d}v\frac{y}{v^2-1}\int_\infty^v\frac{\mathrm{d}v}{y}\int_\infty^v\mathrm{d}v\left(\frac{y}{v^2-1}\right)^3+\mathrm{O}(\varepsilon^3)\right], \tag{16}$$

where Q^σ_ν is the associated Legendre function of second kind and y is taken to be

$$y=\begin{cases}1 & \text{for the ordinary transition}\\ v & \text{for the special transition.}\end{cases} \tag{17}$$

This is the first presentation of the correlation function in real space. The related exponents are given, respectively, by $\eta_\parallel=1+2\mu$ and $\eta_\perp=\frac{1}{2}(1+2\mu+\eta)$ with (14) or (15), which coincide with the previous analysis by Diehl and Dietrich (1981a, b) and Reeve and Guttmann (1981).

Finally, we want to note that the present theory should be valid also in the cases of the extraordinary transition and the anisotropic special transition. Some results using the $1/n$ expansion have been given by Ohno and Okabe (1984) and Ohno *et al* (1985).

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References

Binder K 1983 *Phase Transitions and Critical Phenomena* vol 8, ed C Domb and J L Lebowitz (London: Academic)

Cardy J L 1984 *Nucl. Phys.* B **240** [FS12] 514
Diehl H W and Dietrich S 1981a *Z. Phys.* B **42** 65
—— 1981b *Phys. Rev.* B **24** 2878
Lubensky T C and Rubin M H 1975 *Phys. Rev.* B **11** 4533
Ohno K and Okabe Y 1983 *Prog. Theor. Phys.* **70** 1226
—— 1984 *Prog. Theor. Phys.* **72** 736
Ohno K, Okabe Y and Abe R 1985 *Phys. Lett.* **107A** 41
Okabe Y and Ohno K 1984 *Phys. Rev.* B **30** 6573
Reeve J and Guttmann A J 1981 *J. Phys. A: Math. Gen.* **14** 3357
Polyakov A M 1970 *Zh. Eksp. Teor. Fiz. Pis. Red.* (1970 *JETP Lett.* **12** 381)
Wegner F 1976 *Phase Transitions and Critical Phenomena* vol 6, ed C Domb and M S Green (London: Academic)